ON SUBALGEBRAS OF $n \times n$ MATRICES NOT SATISFYING IDENTITIES OF DEGREE $2n - 2$

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Abstract. The Amitsur-Levitzki theorem asserts that $M_n(F)$ satisfies a polynomial identity of degree $2n$. (Here, $F$ is a field and $M_n(F)$ is the algebra of $n \times n$ matrices over $F$). It is easy to give examples of subalgebras of $M_n(F)$ that do satisfy an identity of lower degree and subalgebras of $M_n(F)$ that satisfy no polynomial identity of degree $\leq 2n - 2$. In this paper we prove that the subalgebras of $n \times n$ matrices satisfying no nonzero polynomial of degree less than $2n$ are, up to $F$-algebra isomorphisms, the class of full block upper triangular matrix algebras.

1. Introduction

This paper is concerned with $n \times n$ matrix subalgebras that do not satisfy a polynomial identity of degree $< 2n$. Our aim is to present and prove the following theorem: Let $F$ be a field and let $A$ be an $F$-subalgebra of $M_n(F)$. If $A$ does not satisfy the standard polynomial $s_{2n-2}$, then $A$ is equivalent to a full block upper triangular matrix algebra.

To begin, let $F$ be a field, $M_n(F)$ the algebra of $n \times n$ matrices over $F$, and $F \{X\} = F \{X_1, X_2, \ldots \}$ the free associative algebra over $F$ in countably many variables. A nonzero polynomial $f(X_1, \ldots, X_m) \in F \{X\}$ is a polynomial identity for an $F$-algebra $R$ (or, $R$ satisfies $f$) if $f(r_1, \ldots, r_m) = 0$ for all $r_1, \ldots, r_m \in R$.

The standard polynomial of degree $t$ is

$$s_t(X_1, \ldots, X_t) = \sum_{\sigma \in S_t} (\text{sgn} \sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(t)},$$

where $S_t$ is the symmetric group on $\{1, \ldots, t\}$ and $(\text{sgn} \sigma)$ is the sign of the permutation $\sigma \in S_t$. The standard polynomial $s_t$ is homogeneous of degree $t$, multilinear and alternating. If $t$ is odd then $s_t(1, X_2, \ldots, X_t) = s_{t-1}(X_2, \ldots, X_t)$. Thus $s_{2t}$ is an identity of $R$ if and only if $s_{2t+1}$ is an identity of $R$. The standard polynomial $s_{q+r}$ is a linear combination, with coefficients being 1 or -1, of evaluations of $s_q s_r$. This can be shown as follows: We partition the set of permutations $S_{q+r}$ by defining the equivalence relation $\tau \sim \sigma$ if the

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images of the interval \([1, q]\) under \(\tau\) and \(\sigma\) are the same set. Then, we have

\[
(1.1) \quad s_t(X_1, \ldots, X_t) = \sum_{\sigma \in S_t/\sim} (\text{sgn} \sigma) s_q(X_{\sigma(1)}, \ldots, X_{\sigma(q)}) s_r(X_{\sigma(q+1)}, \ldots, X_{\sigma(t)}).
\]

The Amitsur-Levitzki theorem asserts that \(M_n(F)\) satisfies any standard polynomial of degree \(2n\) or higher. Moreover, if \(M_n(F)\) satisfies a polynomial of degree \(2n\), then it is a scalar multiple of \(s_{2n}\) (cf. \([AL50]\)).

The standard polynomial \(s_{2n}\) is a minimal identity in the sense that \(M_n(F)\) satisfies no polynomial identity of degree less than \(2n\). More generally, if \(A\) is a subalgebra of \(M_n(F)\) isomorphic to a full block upper triangular matrix algebra,

\[
\begin{pmatrix}
\ast & & & \\
\ast & \ast & & \\
\ast & \ast & \ddots & \\
0 & \ast & \cdots & \ast
\end{pmatrix},
\]

then \(A\) satisfies no polynomial identity of degree less than \(2n\). To prove this assertion, note that every full block upper triangular matrix algebra contains the “staircase sequence” \(e_{11}, e_{12}, e_{22}, e_{23}, \ldots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn}\), and

\[
(1.2) \quad s_{2n-1} \left( e_{11}, e_{12}, e_{22}, e_{23}, \ldots, e_{(n-1)(n-1)}, e_{(n-1)n}, e_{nn} \right) = e_{1n},
\]

where the \(e_{ij}\) are the standard matrix units.

In § 2 we provide the building blocks for the main theorem of this paper and its proof. This proof and some of its consequences are presented in § 3. For polynomial identities in Ring Theory and the polynomial identities of \(n \times n\) matrices, \([Fo91]\) and \([Ro80]\) are suggested general references.

2. Building Blocks

**Lemma 2.1.** Let \(A\) be a simple \(F\)-subalgebra of \(M_n(F)\). Then either \(A = M_n(F)\) or \(A\) satisfies the identity \(s_{2n-2}(A) = 0\).

**Proof.** By assumption, \(A\) is a \(F\)-algebra and central simple algebra over its center \(k\). Let \(K\) denote the algebraic closure of \(k\); then \(A \otimes_k K\) is a simple \(K\)-algebra in a natural way (cf. \([Ro80]\), §1.8), with \(\dim_K (A \otimes_k K) = \dim_k (A)\). Also, \(A \otimes_k K \cong M_t(K)\) for some \(t \leq n\). Suppose that \(A\) is a proper subalgebra of \(M_n(F)\). It follows that \(t < n\). Hence, by the Amitsur-Levitzki theorem, \(A \otimes_k K\) satisfies \(s_{2n-2}\), and the result follows since \(A\) is embedded as a \(k\)-algebra in \(A \otimes_k K\). \(\square\)
2.1. We now consider the case when $A$ contains a "repetition". We will need some notation. (i) Let $M_1, \ldots, M_t$ be matrices in $A$,

$$M_k = \begin{pmatrix} a_k & b_k & c_k \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{pmatrix}, \ a_k \in M_\ell(F), \ e_k \in M_m(F), \ b_k \in M_\ell \times m(F), \ d_k \in M_m \times \ell(F).$$

Given $1 \leq i < j \leq t$ and $\sigma \in S_t$, set

$$m_{\sigma}^\tau[i, j] = (\text{sgn} \sigma) a_{\sigma(1)} \cdots a_{\sigma(i-1)} b_{\sigma(i)} e_{\sigma(i+1)} \cdots e_{\sigma(j-1)} d_{\sigma(j)} a_{\sigma(j+1)} \cdots a_{\sigma(t)},$$

and denote by $W$ the set of all matrix products

$$\{m_{\sigma}^\tau[i, j] : \sigma \in S_t \text{ and } 1 \leq i < j \leq t\}.$$

(ii) The projection $ur$ returns the $\ell \times \ell$ upper right block of a matrix in $A$:

$$ur \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} = c.$$

(iii) Given $n \times n$ matrices $M_1, \ldots, M_t$, we say that a matrix product $M_1 \cdots M_t$ formally contains the factor $A_1 \cdots A_s$ if $A_1 = M_1, A_2 = M_{\ell+1}, \ldots, A_s = M_{\ell+s-1}$, for some $1 \leq \ell \leq t$. This notation is to distinguish to the case when $CA_1 \cdots A_sD = M_1 \cdots M_t$ as $n \times n$ matrices, for some matrices $C$ and $D$. Further, if $\ell = 1$, we say that $M_1 \cdots M_t$ formally contains $A_1 \cdots A_s$ as left factor.

This is a good place to record a Lemma extracted from [AL50], which will be used later.

**Lemma 2.2.** [AL50, Lemma 1, 450-451] If for an odd positive integer $r$ we put $Y = X_{i+1} \cdots X_{i+r}$, and if $s'$ denotes the sum of all terms of $s_m(X)$ containing the common factor $Y$, then

$$s' = s_{m-r+1}(X_1, \ldots, X_i, Y, X_{i+r+1}, \ldots, X_m).$$

**Lemma 2.3.** Set $t = 2(\ell + m)$, and let $M_1, \ldots, M_t$ be matrices in $A$ such that for all $1 \leq k \leq t$,

$$M_k = \begin{pmatrix} a_k & b_k & 0 \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{pmatrix}, \ a_k \in M_\ell(F), \ e_k \in M_m(F), \ b_k \in M_\ell \times m(F), \ d_k \in M_m \times \ell(F).$$

Then $ur[s_t(M_1, \ldots, M_t)] = 0.$
First we observe that

\[ ur[M_1 \cdots M_t] = \sum_{1 \leq i < j \leq t} a_{\sigma(1)} \cdots a_{\sigma(i-1)} b_{\sigma(i)} e_{\sigma(i+1)} \cdots e_{\sigma(j-1)} d_{\sigma(j)} a_{\sigma(j+1)} \cdots a_{\sigma(t)}, \]

which implies that

\[ ur [s_t(M_1, \ldots, M_t)] = \sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t} m^\sigma_t[i, j]. \] (2.3)

To prove that \( ur [s_t(M_1, \ldots, M_t)] = 0 \), we split the right hand side into two summands:

\[ ur [s_t(M_1, \ldots, M_t)] = \]

\[ \sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t} m^\sigma_t[i, j] + \sum_{\sigma \in S_t} \sum_{j-i-1 \geq 2m} m^\sigma_t[i, j] \] (2.4)

Our goal is to show that each summand in (2.4) is zero. To handle the first summand we introduce the following new equivalence relation on \( S_t \). Given fixed \( 1 \leq i < j \leq t \), such that \( j-i-1 \geq 2m \), and given \( \tau, \sigma \in S_t \), say that \( \tau \) is \([i, j]\)-equivalent to \( \sigma \) if \( \tau \) restricted to the initial and final intervals \([1, i]\) and \([j, t]\) equals the restriction of \( \sigma \) to the same domain. In symbols,

\[ \tau \sim_{[i, j]} \sigma \iff \tau|_{[1, i]} = \sigma|_{[1, i]} \text{ and } \tau|_{[j, t]} = \sigma|_{[j, t]} \]

For each pair \( i, j \), such that \( 1 \leq i < j \leq t \) and \( j-i-1 \geq 2m \), the relation \( \sim_{[i, j]} \) yields a partition of \( S_t \) into disjoint subsets \( P^k_{[i, j]} = 1, \ldots, \frac{t}{j-i-1} \). Then, we have

\[
\sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t \atop j-i-1 \geq 2m} m^\sigma_t[i, j] = \sum_{\sigma \in S_t} \sum_{k \in P^k_{[i, j]}} \sum_{1 \leq i < j \leq t \atop j-i-1 \geq 2m} m^\sigma_t[i, j] = \\
\sum_{1 \leq i < j \leq t \atop j-i-1 \geq 2m} \sum_{k} \sum_{\sigma \in P^k_{[i, j]}} (sg\sigma) a_{\sigma(1)} \cdots a_{\sigma(i-1)} b_{\sigma(i)} e_{\sigma(i+1)} \cdots e_{\sigma(j-1)} d_{\sigma(j)} a_{\sigma(j+1)} \cdots a_{\sigma(t)} \\
= \\
\sum_{1 \leq i < j \leq t \atop j-i-1 \geq 2m} \sum_{k} (sg\sigma_k) a_{\sigma_k(1)} \cdots a_{\sigma_k(i-1)} b_{\sigma_k(i)} \sum_{j-i-1 \geq 2m} d_{\sigma_k(j)} a_{\sigma_k(j+1)} \cdots a_{\sigma_k(t)}; \\
\]

where \( s = s_{i-j+1}(e_{\sigma_k(i+1)}, \ldots, e_{\sigma_k(j-1)}) \) and \( \sigma_k \) is a representative of the class \( P^k_{[i, j]} \). Since \( j-i-1 \geq 2m \),

\[ s_{i-j+1}(e_{\sigma_k(i+1)}, \ldots, e_{\sigma_k(j-1)}) = 0 \quad \text{for all } k, \]
hence

\[
\sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t} m_1^\sigma[i, j] = 0.
\]

This takes care of the first term in (2.4). We now turn to the second summand. For a given \( q \), with \( 2 \leq q \leq t \), denote by \( R_q \) the set of all \( q \)-tuples \( r = (r_1, \ldots, r_q) \) of different elements from \( \{1, \ldots, t\} \) and by \( T_{(r_1, \ldots, r_q)} \) the set of matrix products \( w \) formally containing the common factor \( b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \). Considering all possible \( q \) and \( q \)-tuples, the sets \( T_{(r_1, \ldots, r_q)} \) form a partition of \( W \). We are interested in the case when \( q \leq 2m + 1 \). Observe that

\[
\sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t} m_1^\sigma[i, j] = \sum_{q=2}^{2m+1} \sum_{r \in R_q} \sum_{w \in T_{(r_1, \ldots, r_q)}} w.
\]

Fix \( q \) odd, a \( q \)-tuple \( (r_1, \ldots, r_q) \), and the corresponding set of matrix products \( T_{(r_1, \ldots, r_q)} \). Then, \( \sum_{w \in T_{(r_1, \ldots, r_q)}} w \) is the sum of all matrix products formally containing the common factor \( y = b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \). Each matrix product \( w \in T_{(r_1, \ldots, r_q)} \) corresponds uniquely to a permutation \( \sigma \in S_t \) and a pair \((i, j)\), such that the \( q \)-tuple \((r_1, \ldots, r_q)\) is the image under \( \sigma \) of \((i, \ldots, j)\). Explicitly, the correspondence is \( w = m_1^\sigma[i, j] \). We can now apply Lemma 2.2 and the alternating property of the standard polynomials. If \( \sigma_0 \in S_t \) is a fixed permutation such that

\[
\sigma_0(i) \mapsto r_i, \ 1 \leq i \leq q,
\]

we have

\[
\sum_{w \in T_{(r_1, \ldots, r_q)}} w = (sg \sigma_0) s_{t-q+1} (y, a_{\sigma_0(q+1)}, \ldots, a_{\sigma_0(t)})
\]

where \( y = b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \). Since \( t - q + 1 \geq 2\ell \), and since all the arguments of \( s_{t-q+1} \) in the last equation are \( \ell \times \ell \) matrices, it follows that

\[
(2.5) \sum_{w \in T_{(r_1, \ldots, r_q)}} w = 0, \text{ when } q \text{ is odd and } (r_1, \ldots, r_q) \text{ is a fixed } q \text{-tuple.}
\]

Therefore

\[
\sum_{q=2}^{2m+1} \sum_{q \text{ odd}} \sum_{r \in R_q} \sum_{w \in T_{(r_1, \ldots, r_q)}} w = 0.
\]

Suppose now that \( q \) is even, so \( q \leq 2m \), and fix an arbitrary \( q \)-tuple \( r = (r_1, \ldots, r_q) \). We will split further the sets \( T_r \). First consider all \( w \in T_r \) formally containing in common the left factor \( y = b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \), and call this subset \( L_r \). Then, for each \( r_0 \notin \{r_1, \ldots, r_q\} \) consider the \((q+1)\)-tuple \((r_0, r)\) and the subset \( G_{(r_0, r)} \) of \( w \in T_r \) formally containing in
common the factor \( y = a_{r_0} b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \). The sum of all matrix products in the set \( T_r \) can be split as

\[
\sum_{w \in T_r} w = \sum_{w \in L_r} w + \sum_{r_0 : r_0 \neq r_1, \ldots, r_q} \sum_{w \in G_{(r_0, r)}^q} w.
\]

For the terms in \( L_r \) we have

\[
(2.6) \quad \sum_{w \in L_{(r_1, \ldots, r_q)}} w = (s g \sigma_0) y s t - q \left( a_{\sigma_0(0)}, \ldots, a_{\sigma_0(t)} \right),
\]

where \( y = b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \), and where \( \sigma_0 \in S_t \) is a fixed permutation such that

\[
\sigma_0 : i \to r_i, \quad 1 \leq i \leq q.
\]

Since \( t - q \geq 2\ell \), we obtain

\[
(2.7) \quad \sum_{w \in L_r} w = 0.
\]

Finally, for a suitable fixed \( r_0 \), the sequence \((r_0, r)\) has odd length, so we can argue as in (2.5) to obtain

\[
\sum_{w \in G_{(r_0, r)}} w = (s g \sigma_0) s t - q + 1 \left( y, a_{\sigma_0(0)}, \ldots, a_{\sigma_0(t)} \right) = 0,
\]

where \( y = a_{r_0} b_{r_1} e_{r_2} \cdots e_{r_{q-1}} d_{r_q} \), and where \( \sigma_0 \in S_t \) is a fixed permutation such that

\[
\sigma_0 = \begin{cases} 
1 \to r_0, \\
i \to r_{i-1}, & \text{for } 2 \leq i \leq q + 1.
\end{cases}
\]

This finishes the proof of Lemma 2.3. \( \square \)

**Proposition 2.4.** Let

\[
A = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} : a, c \in M_\ell(F), e \in M_m(F), b \in M_\ell \times M_m(F), d \in M_{m \times \ell}(F) \right\}.
\]

Then, \( A \) satisfies \( s_{2(\ell + m)} \).

**Proof.** For any \( t \) and matrices \( M_k \in A, k = 1 \ldots t \), set

\[
M_k = \begin{pmatrix} a_k & b_k & c_k \\ 0 & e_k & d_k \\ 0 & 0 & a_k \end{pmatrix}, \quad a_k \in M_\ell(F), c_k \in M_m(F), b_k \in M_\ell \times M_m(F), d_k \in M_{m \times \ell}(F).
\]
By direct calculations, we obtain
\[
ur[s_t(M_1, \ldots, M_t)] = \\
= \sum_{i=1}^{t} s_t(a_1, \ldots, a_{i-1}, c_i, a_{i+1}, \ldots, a_t) + \sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t} m_{\sigma}^t[i, j].
\]

Now set \( t = 2(\ell + m) \). It follows from (2.3) that
\[
\sum_{\sigma \in S_t} \sum_{1 \leq i < j \leq t} m_{\sigma}^t[i, j] = ur[s_t(M_1', \ldots, M_t')] = 0,
\]
where \( M_k' \) is the matrix in \( A \) obtained by replacing the upper right corner \( c_k \) of \( M_k \) by \( 0 \in M_\ell(F) \). Suitable applications of the Amitsur-Levitzki identity give us
\[
ur[s_t(M_1, \ldots, M_t)] = 0,
\]
and
\[
s_t \begin{pmatrix} a_1 & b_1 \\ 0 & e_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_t & b_t \\ 0 & e_t \end{pmatrix} = 0,
\]
and
\[
s_t \begin{pmatrix} e_1 & d_1 \\ 0 & a_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_t & d_t \\ 0 & a_t \end{pmatrix} = 0.
\]
Combining these three equations, it follows that \( s_t(M_1, \ldots, M_t) = 0 \). \( \square \)

3. Main Theorem

In this section we prove that if an \( F \)-subalgebra of \( M_n(F) \) does not satisfy the standard polynomial \( s_{2n-2} \), then it is isomorphic as \( F \)-algebra to a full block upper triangular matrix algebra.

3.1. We first introduce our notation and review some necessary background (cf. \[Le02\]).

(i) Let \( t \) be a positive integer, let \( \ell_1, \ell_2, \ldots, \ell_t \) be positive integers summing up to \( n \), and set
\[
E_{(\ell_1, \ell_2, \ldots, \ell_t)}(F) = \begin{pmatrix} M_{\ell_1}(F) & M_{\ell_1 \times \ell_2}(F) & \cdots & M_{\ell_1 \times \ell_{t-1}}(F) & M_{\ell_1 \times \ell_t}(F) \\ 0 & M_{\ell_2}(F) & \cdots & M_{\ell_2 \times \ell_{t-1}}(F) & M_{\ell_2 \times \ell_t}(F) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{\ell_{t-1}}(F) & M_{\ell_{t-1} \times \ell_t}(F) \\ 0 & 0 & \cdots & 0 & M_{\ell_t}(F) \end{pmatrix},
\]
a full block upper triangular matrix subalgebra of \( M_n(F) \).

(ii) Recall that every \( F \)-algebra automorphism \( \tau \) of \( M_n(F) \) is inner (i.e., there exists an invertible \( Q \) in \( M_n(F) \) such that \( \tau(a) = QaQ^{-1} \) for all \( a \in M_n(F) \)). We will say that two \( F \)-subalgebras \( A, A' \) of \( M_n(F) \) are equivalent provided there exists an automorphism \( \tau \) of \( M_n(F) \) such that \( \tau(A) = A' \).
(iii) We will say that a subalgebra $\Lambda$ of $E_{(\ell_1, \ell_2, \ldots, \ell_t)}(F)$ is an $(\ell_1, \ell_2, \ldots, \ell_t)$-extension of simple blocks if the projections $\pi_i : \Lambda \to M_{\ell_i}(F)$, for $1 \leq i \leq t$, are all irreducible representations (when $F$ is algebraically closed, of course, the representation $\pi_i$ is irreducible if and only if $\pi_i(\Lambda) = M_{\ell_i}$). Note that, every $F$-subalgebra $A$ of $M_n(F)$ is equivalent to an $(\ell_1, \ell_2, \ldots, \ell_t)$-extension of simple blocks $\Lambda$ for some suitable $(\ell_1, \ell_2, \ldots, \ell_t)$.

**Theorem 3.1.** Let $F$ be a field and let $A$ be an $F$-subalgebra of $M_n(F)$. If $A$ does not satisfy the standard polynomial $s_{2n-2}$, then $A$ is equivalent to a full block upper triangular matrix algebra.

*Proof.* Without loss of generality we assume that $A$ is an extension of simple blocks. We proceed by induction on $t$, the number of diagonal blocks in $A$. If $t = 1$, $A$ is a simple algebra and therefore, in view of Lemma 2.1, $A$ is a full matrix algebra. Now suppose that there are $t$ diagonal blocks of sizes $\ell_1, \ell_2, \ldots, \ell_t$, with $n = \ell_1 + \cdots + \ell_t$. The desired conclusion is: If $A$ does not satisfy $s_{2n-2}$, then it is full block upper triangular. Assume that each matrix $B \in A$ has the form

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1(t-1)} & B_{1t} \\ 0 & B_{22} & \cdots & B_{2(t-1)} & B_{2t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{(t-1)(t-1)} & B_{(t-1)t} \\ 0 & 0 & \cdots & 0 & B_{tt} \end{pmatrix}.$$  

If $A$ does not satisfy $s_{2n-2}$, then it does not satisfy $s_{2(\ell_1+\ell_2+\cdots+\ell_{t-1})-2} s_{2t}$. This implies that the image of $A$ gotten by deleting the last row and the last column does not satisfy $s_{2(\ell_1+\ell_2+\cdots+\ell_{t-1})-2}$, and by induction it is full block upper triangular. Similarly, the image of $A$ gotten by deleting the first row and the first column is full block upper triangular. Since $A$ can not be semisimple, $A$ is full block upper triangular unless the projections $B \to B_{11}$ and $B \to B_{tt}$ are equivalent representations of $A$, which means that there is a fixed matrix $T$ such that $T B_{11} T^{-1} = B_{tt}$ for all $B \in A$. But then Proposition 2.4 implies that $A$ satisfies $s_{2n-2}$. \hfill $\Box$

**Corollary 3.2.** The standard polynomial $s_{2n-2}$ is an identity for any proper subalgebra of $U_n(F)$, the algebra of upper triangular matrices over the field $F$.

*Proof.* Immediate from Theorem 3.1. \hfill $\Box$

**Remark.** The standard polynomial of degree $2n - 2$ is not necessarily an identity for any proper subalgebra of $U_n(C)$ when $C$ is a commutative ring: Let $I$ be a nonzero ideal of $C$, and consider the $C$-subalgebra $B$ of $U_n(C)$ defined by the property that the $(1,2)$-entry of matrices in $B$ lie in $I$. A staircase argument shows that $s_{2n-2}(B) \neq 0.$
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